Computational Modeling of Wave Propagation in a Geophysical Domain

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Abstract: The propagation of shear (S) and compression (P) waves within the earth allows geologists to track seismic events and to identify subterranean structure. Highly specialized geological based computer programs developed have been instrumental in determining the location and characteristics of natural phenomena (e.g., earthquakes) and man-made activity (e.g., nuclear-blast tests). Use of these internally developed programs requires regular maintenance, and the reliance on supercomputers limits broad accessibility. This paper seeks to demonstrate that commercially available software running on desktop computational resources can provide accurate solutions to an important subset of problems associated with wave propagation in the geophysical domain. The work presented here uses COMSOL Multiphysics to solve the equilibrium equations for a time-varying system using the finite element method. This work focuses on developing a benchmark solution of a homogeneous half-space loading with an impact and develops a general closed-form solution against which to compare the computational results. These results show the ability to resolve both *S* and *P* wave across the computational domain. Thus, COMSOL Multiphysics running on desktop computational resources provides sufficiently accurate results for critical geophysical wave propagation problems.

Keywords: Geophysics, shear wave, pressure wave, seismic.

1. Introduction

Scientists and engineers that seek to understand elastic wave propagation in geological structures typically consider the generic problem of a seismic wave that is generated at a source, propagates through a media, and is measured at a receiver. Some researchers are concerned with natural sources, e.g. earthquakes, while others focus on man-made sources, e.g. explosions. Some who study these areas seek to understand the nature of the source, while others use well characterized sources and seek to characterize the medium. These researchers may be seeking to predict future earthquakes, locate natural resources, or identify and locate specific human activities. However, in all these cases, the overarching physics of elastic wave propagation in a solid medium remain the same.

Seismic waves propagating in bulk material do so as either compressional waves, *P*, where material translates in the direction of wave propagation, and shear waves, *S*, where material translates perpendicular to the direction of wave propagation. *P*-waves travel faster than *S*-waves. Another important class of waves is surface waves; these waves develop due to an energy concentration near the Earth's surface and consequently propagate in two dimensions. As such, surface waves decay as $1/r$ while body waves decay as $1/r^2$ due to their propagation in three-dimensions. Thus, energy measured at a near-surface receiver located a significant distance from the source is dominated by surface waves. Typically, these types of waves are referred to as either *Love* or *Rayleigh waves*. *Love waves* are shear waves that have been polarized in the horizontal direction (parallel to the Earth's surface) while *Rayleigh waves* are a mixture of *P* waves and *S* waves that have been polarized in the vertical direction.

Geophysics studies typically use specialized computer software running on supercomputers to simulate wave propagation in geophysical domains. This work demonstrates that a finite element based software package that is commercially available, COMSOL Multiphysics, solves these wave propagation problems using readily available desktop computational hardware.

Report Documentation Page

Form Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
OCT 2008		**00-00-2008 to 00-00-2008**

4. TITLE AND SUBTITLE
Computational Modeling of Wave Propagation in a Geophysical Domain

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
US Army Engineering Research and Development Center,Geotechnical & Structures Laboratory,3909 Halls Ferry Road,Vicksburg,MS,39180

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES
Presented at the 2008 COMSOL Conference Boston, 9-11 Oct, Boston, MA

14. ABSTRACT
The propagation of shear (S) and compression (P) waves within the earth allows geologists to track seismic events and to identify subterranean structure. Highly specialized geological based computer programs developed have been instrumental in determining the location and characteristics of natural phenomena (e.g., earthquakes) and man-made activity (e.g., nuclear-blast tests). Use of these internally developed programs requires regular maintenance, and the reliance on supercomputers limits broad accessibility. This paper seeks to demonstrate that commercially available software running on desktop computational resources can provide accurate solutions to an important subset of problems associated with wave propagation in the geophysical domain. The work presented here uses COMSOL Multiphysics to solve the equilibrium equations for a time-varying system using the finite element method. This work focuses on developing a benchmark solution of a homogeneous half-space loading with an impact and develops a general closed-form solution against which to compare the computational results. These results show the ability to resolve both S and P wave across the computational domain. Thus, COMSOL Multiphysics running on desktop computational resources provides sufficiently accurate results for critical geophysical wave propagation problems.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			
unclassified	**unclassified**	**unclassified**	**Same as Report (SAR)**	**21**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

2. Method

This work uses the Structural Mechanics Module available in COMSOL Multiphysics to develop computational models for a range of problems.

These models increase with complexity ranging from a simple point source in an infinite solid to a volume source that represents experimentally measure forces applied in a layered half-space.

2.1 Point Source in an Infinite Media

Initially, this work focuses on modeling a point source located in an infinite domain. To provide a point of comparison for the finite element models, an analytical solution for this problem was developed, and is described as follows. Using the notation u_{ij} to represent the displacement in the $i-$direction due to a concentrated point force, specified as $\mathbf{f}(\mathbf{x},t;\boldsymbol{\xi}) = f_0(t)\delta(\mathbf{x}-\boldsymbol{\xi})\mathbf{e}_j$, applied in the $j-$direction at the point $\boldsymbol{\xi}$, the displacements over the domain may be determined by solving the following equation;

$$4\pi\rho\, u_{ij}(\mathbf{x},t) = \frac{(3\gamma_i\gamma_j - \delta_{ij})}{r^3}\int_{r/\alpha}^{r/\beta}\tau\, f_0(t-\tau)d\tau + \frac{\gamma_i\gamma_j}{\alpha^2 r} f_0\left(t-\frac{r}{\alpha}\right) - \frac{(\gamma_i\gamma_j - \delta_{ij})}{\beta^2 r} f_0\left(t-\frac{r}{\beta}\right)$$

where

ρ is the material density,

$r = |\mathbf{x}-\boldsymbol{\xi}|$ is the distance from the source

$\gamma_i = \dfrac{x_i - \xi_r}{r}$ are the direction cosines of vector, and

$$\alpha = c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

and

$$\beta = c_S = \sqrt{\frac{\mu}{\rho}}:$$

represent the speeds of pressure and shear waves, respectively. In these equations, λ, μ are Lame elastic constants,

This work represents the typical source force as: $f_0(t) = H(t) = a^2 t e^{-at}$, where $H(t)$ is the Heaviside step function, a is a parameter controlling duration and amplitude of the source, for additional information see [1].

Three finite element models of a spherical domain loaded by a point source were developed in COMSOL Multiphysics. The volume was represented in three-dimensions using a quarter symmetric finite element model. To reduce the computational cost of this model, two-dimensions models where constructed using an axisymmetric and a plane strain formulation.

2.2 Point Source in a Semi-Infinite Media

To extend this work to include surface wave, a solution to Lamb's problem of a source loading at the surface of a semi-infinite domain is developed. Again, a closed form solution was developed as a point of comparison for the finite element solution of this problem. In this closed form solution, the vertical stress acting on the free surface is denoted as σ_{33};

where

$f(t) = \pi a^2 \sigma_{33}(t)$ denotes the total applied force

and

a denotes the radius of the circle over which the stress is applied.

From this loading, the vertical displacement is denoted $w(t)$ and horizontal displacements are denoted $u(t)$. Thus, the solution for a special case of Lamb's problem may be written as

$$\begin{Bmatrix} w(t) \\ u(t) \end{Bmatrix} = \frac{\sigma_{33}}{\pi^2 \mu r}\left(\frac{\alpha}{\beta}\right)^2 \frac{r}{\beta}\int_{-\infty}^{\tau} \left.\frac{df}{dt}\right|_{t=r\tau'/\beta} \begin{Bmatrix} G(\tau-\tau') \\ R(\tau-\tau') \end{Bmatrix} d\tau'$$

where

$$G(\tau)=\begin{cases}0, \quad \tau<1/\delta \\ -\dfrac{\pi}{96}\left[6-\dfrac{\left(3\sqrt{3}+5\right)^{1/2}}{\left(\gamma^2-\tau^2\right)^{1/2}}+\dfrac{\left(3\sqrt{3}-5\right)^{1/2}}{\left(\tau^2+\sqrt{3}/4-3/4\right)^{1/2}}-\dfrac{\sqrt{3}}{\left(\tau^2-1/4\right)^{1/2}}\right], \quad 1/ \\ -\dfrac{\pi}{48}\left[6-\dfrac{\left(3\sqrt{3}+5\right)^{1/2}}{\left(\gamma^2-\tau^2\right)^{1/2}}\right], \quad 1<\tau<\gamma \\ -\pi/8, \quad \tau>\gamma\end{cases}$$

$$R(\tau)=\begin{cases}0, \quad \tau<1/\delta \\ \dfrac{\tau}{16\sqrt{6}}\left\{\begin{aligned}&6\mathrm{K}(k)-18\Pi\left(8k^2,k\right)+\left(6-4\sqrt{3}\right)\Pi\left[\left(20-12\sqrt{3}\right)k^2,k\right]\\&+\left(6+4\sqrt{3}\right)\Pi\left[\left(20+12\sqrt{3}\right)k^2,k\right]\end{aligned}\right\}, \quad 1/ \\ \dfrac{\tau/k}{16\sqrt{6}}\left\{\begin{aligned}&6\mathrm{K}\left(\frac{1}{k}\right)-18\Pi\left(8,\frac{1}{k}\right)+\left(6-4\sqrt{3}\right)\Pi\left[\left(20-12\sqrt{3}\right),\frac{1}{k}\right]\\&+\left(6+4\sqrt{3}\right)\Pi\left[\left(20+12\sqrt{3}\right),\frac{1}{k}\right]\end{aligned}\right\}, \quad 1< \\ \text{Preceding}+\dfrac{\pi\tau}{24}\left(\tau^2-\gamma^2\right)^{-1/2}, \quad \tau>\gamma\end{cases}$$

and,

$\delta^2=\alpha/\beta$, $\gamma=\left(3+\sqrt{3}\right)^{1/2}/2$,

$\tau=(\beta/r)t$,

$k^2=\left[(\alpha/\beta)^2\tau^2-1\right]/\left[(\alpha/\beta)^2-1\right]$.

The $\mathrm{K}(k)$ and $\Pi(n,k)$ are complete elliptic integrals of the first type and third types, respectively.

This solution proves valid for Poisson's ratio of $\nu=0.25$. A solution for arbitrary ν is given in [2]

An axisymmetric model of a semi-infinite domain was developed in COMSOL Multiphysics to solve Lamb's problem. For both the closed-form and finite element solution, the forcing function is described as

$$f(t)=h\cos^2\left(\frac{\pi}{T}t\right) \text{ for } -\frac{T}{2}\le t\le\frac{T}{2}.$$

For the finite element analyses conducted in this work the period of the loading is 10 μs, ($T=10\mu s$).

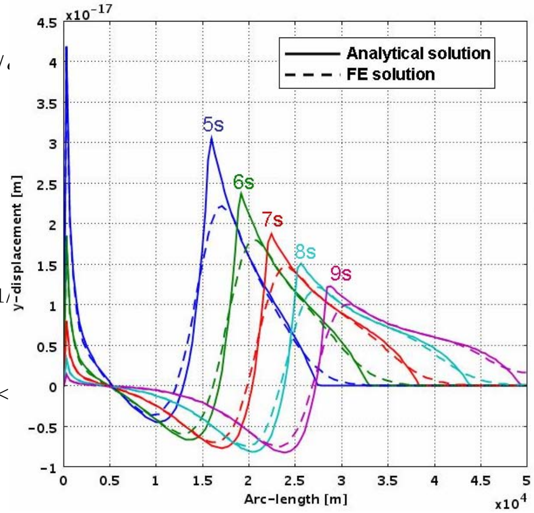

Figure 1. Comparison of vertical displacements for a three-dimensional finite element solution to a closed-form solution for point loading in an infinite domain.

2.3 Comparison with Experimental Data

The next level of complexity for this work is to represent an actual geophysical domain over which experimental data was taken. The model developed previously in COMSOL Multiphysics was modified to include measured pressure wave speeds (c_p) and shear wave speeds (c_s) and to represent the actual forcing function applied the surface. Over a depth of thirty meters, ten material layers that represent experimentally measured values of c_p and c_s values were included in the model. The pressure wave velocity increased from approximately 650 m/s to 2000 m/s with increasing depth. The shear wave velocities increased from 200 m/s to 600 m/s over this range of depths.

3. Discussion

3.1 Point Source in an Infinite Media

Error! Reference source not found. shows the comparison of vertical displacements over a radial line from the center of the sphere. The results show the transmission of the wave as it moves through the infinite domain. These results show the ability of COMSOL Multiphysics to represent accurately the wave except for minor variations at the peak of the wave. These differences may be due to the mesh size used in this model.

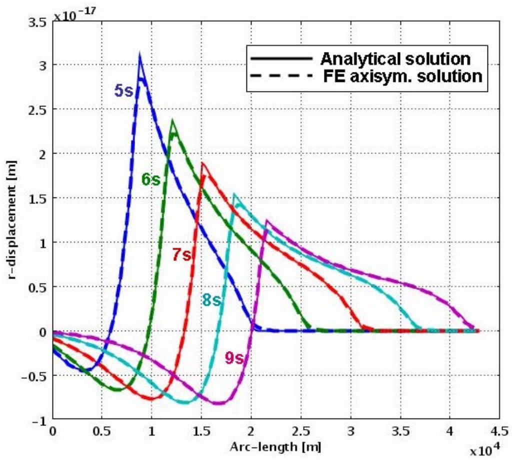

Figure 2. Comparison of radial displacements for a three-dimensional finite element solution to closed-form solution for point loading in an infinite domain.

Figure 2 compares the results developed for an axisymmetric model of an infinite domain. These results show similar accuracy compared to the three-dimension results shown in **Error! Reference source not found.**. Near the peak of the wave, the axisymmetric model shows greater accuracy than the three-dimensional model. This improved accuracy may be due to the small finite element size used in the axisymmetric models.

This work also used a plane strain formulation to assess the ability to represent a point source using a plane strain formulation. Figure 3 clearly demonstrates that the plane strain formulation provides a poor approximation to a point source. The plane strain formulation transforms a point source into a line source. Thus, the computed wave form changes dramatically between the three-dimensional and plane strain solutions. These results indicate that an axisymmetric formulation provides a better method to reduce the computational cost of this problem than a plane strain formulation.

3.2 Point Source in a Semi-Infinite Media

Figure 4 compares results developed from a finite element analysis conducted using COMSOL Multiphysics with the analytical solution developed in Section 2.2. These results show the ability of the finite element analysis to represent accurately the wave propagation generated from a volume source through a semi-infinite domain. The finite element results show the lower magnitude, faster moving *P* wave that precedes the *Rayleigh* wave. The finite element analyses accurately calculate the arrival time and frequency content of this pulse. However, the finite element analyses under predict the magnitude of the velocity by approximately 20%.

3.3 Effects of Model Details

To compare with experimental data, a finite element model was constructed that includes the variation of the soil properties as described in Section 2.3. In Figure 5, the effects of the variable wave speed are shown compared to a model that has homogenous material properties equal to the first layer in the variable model.

The homogeneous model shows clearly defined *P* and *Rayleigh* waves while the layered model shows the wave reflections that developed due to the variations in material properties through the depth of the model. The shorter arrival time for the layered model represents one interesting feature of this comparison: the increased wave speed below the surface of the model generates a wave that reflects off a subsurface layer and arrives back at the surface in less time than the *P* wave traveling in the first layer of the model.

3.4 Comparison with Experimental Data

Figure 6 compares the radial velocity measured during a well-controlled series of experiments with results from the finite element analyses. The magnitude of the velocities has been normalized by the peak magnitude during the duration of the pulse. This normalization was done to provide a comparison using the two primary metrics of interest in this work: arrival times and frequency content. Prediction of the magnitude of the wave is a secondary consideration. These results show a reasonable agreement between the finite element results and experimental data for the primary factors of interest.

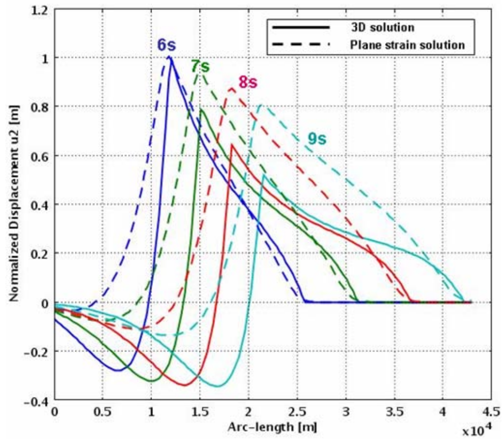

Figure 3. Comparison of vertical displacements for three-dimensional and plane strain finite element solution.

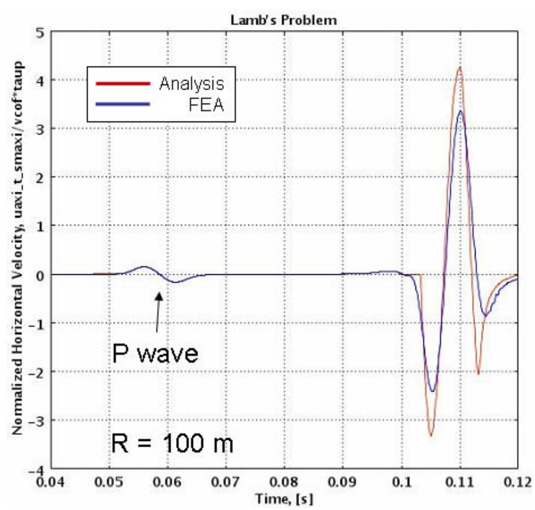

Figure 4. Comparison of analytical solution and finite element solution for a near-surface point located at 100 m from source ($T = 10\mu s$).

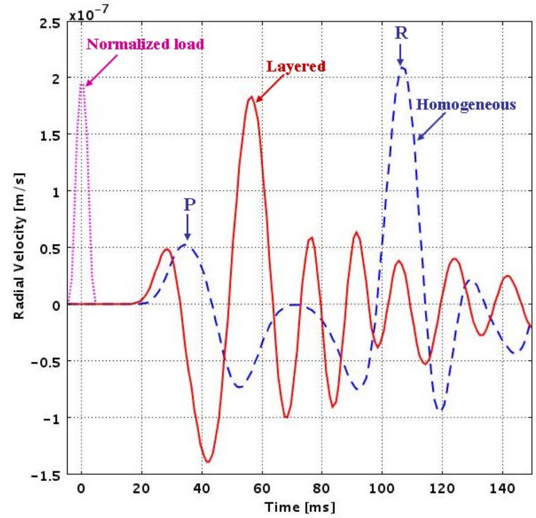

Figure 5. Effect of varying wave speed through layers in the computational model (R=20 m).

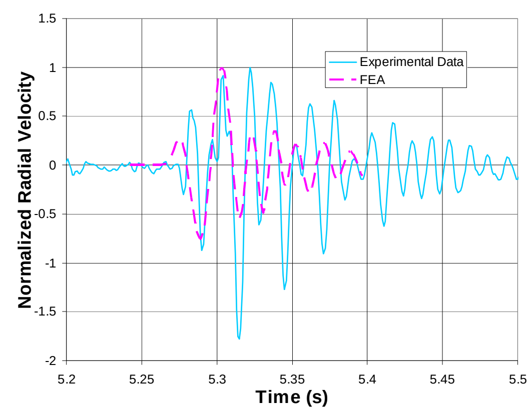

Figure 6. Comparison of finite element results with experimental data (R=20 m). The time specified on the ordinate is relative to the start of the experiment.

4. Summary and Conclusions

The results presented in this paper demonstrate the capabilities of a COMSOL Multiphysics to solve wave propagation problems in finite geophysical domains. The analyses were conducted using computational hardware that is readily available on the desktop. To maximize the use of desktop hardware, this work examines methods for reducing the problem size. For the point source problems considered here, axisymmetric modeling provides significantly more accurate results compared with plane strain modeling techniques.

This work initially focuses on solving problems in a uniform infinite space or infinite half-space for which analytical solutions exist. After showing good agreement with these solutions, additional complexity is added to represent available experimental data. The modeling methods developed in this work again show good agreement with data developed. This work shows the strong effect of including the variation of wave speed through the top thirty meters of the earth. By including this variation in material properties, the wave arrival time and frequency content agree with experimental data. Thus, this work demonstrates that COMSOL Multiphysics provides a useful tool for predicting wave propagation using commercially available finite element software with desktop computing hardware.

8. References

1. Pujol, J., *Elastic Wave Propagation and Generation in Seismology*, Cambridge University Press, 2003.

2. Mooney, H. M., “Some Numerical Solutions for Lamb’s Problem,” *Bulletin of Seismological Society of America*, Vol. 64, No. 2, pp. 473-491, April 1974

Presented at the COMSOL Conference 2008 Boston

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Overview

- Objective:

  Demonstrate the capability of COMSOL Multiphysics to accurately solve wave propagation problems in geophysics
- Motivation:

  Reduce reliance on custom software and supercomputers by obtaining solution using commercially available software on high-end desktop computer
- Approach:
  - Develop closed-form solutions for
    - Point source in an infinite body
    - Point source on the surface of a semi-infinite body (Lamb's problem)
  - Develop using solid mechanics module w/ COMSOL
    - Same formulation as acoustics module
    - Three-dimensional
    - Axisymmetric
    - Plane strain
  - Comparison w/ experimental data
    - Hammer blow on surface

Closed Form Solution - Displacement

- Elastic Wave in an Infinite Body

$$4\pi\rho \; u_{ij}(\mathbf{x},t) = \frac{(3\gamma_i\gamma_j - \delta_{ij})}{r^3}\int_{r/\alpha}^{r/\beta} \tau \; f_0(t-\tau)d\tau + \frac{\gamma_i\gamma_j}{\alpha^2 r} f_0\left(t - \frac{r}{\alpha}\right) - \frac{(\gamma_i\gamma_j - \delta_{ij})}{\beta^2 r} f_0\left(t - \frac{r}{\beta}\right)$$

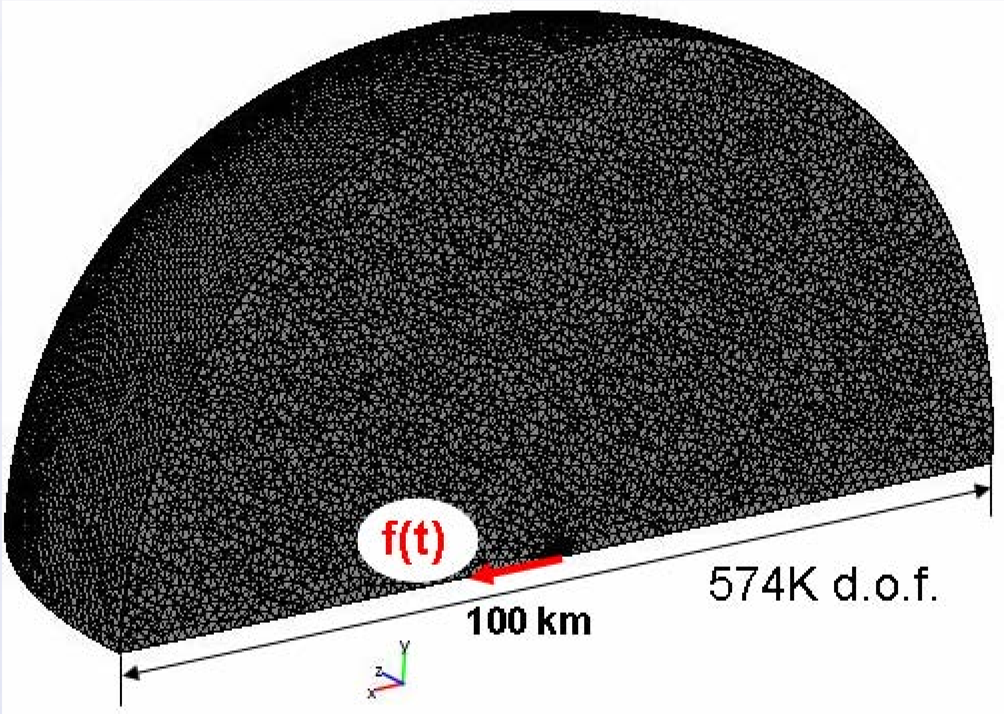
- Semi-Infinite Body (Lamb's Problem- Fixed Poisson ratio)

$$\begin{Bmatrix} w(t) \\ u(t) \end{Bmatrix} = \frac{\sigma_{33}}{\pi^2 \mu r}\left(\frac{\alpha}{\beta}\right)^2 \frac{r}{\beta}\int_{-\infty}^{\tau} \left.\frac{df}{dt}\right|_{t=r\tau'/\beta} \begin{Bmatrix} G(\tau-\tau') \\ R(\tau-\tau') \end{Bmatrix} d\tau'$$

$$G(\tau) = \begin{cases} 0, \quad \tau < 1/\delta \\ -\dfrac{\pi}{96}\left[6 - \dfrac{(3\sqrt{3}+5)^{1/2}}{(\gamma^2-\tau^2)^{1/2}} + \dfrac{(3\sqrt{3}-5)^{1/2}}{(\tau^2+\sqrt{3}/4-3/4)^{1/2}} - \dfrac{\sqrt{3}}{(\tau^2-1/4)^{1/2}}\right], \quad 1/\delta < \tau < 1 \\ -\dfrac{\pi}{48}\left[6 - \dfrac{(3\sqrt{3}+5)^{1/2}}{(\gamma^2-\tau^2)^{1/2}}\right], \quad 1 < \tau < \gamma \\ -\pi/8, \quad \tau > \gamma \end{cases}$$

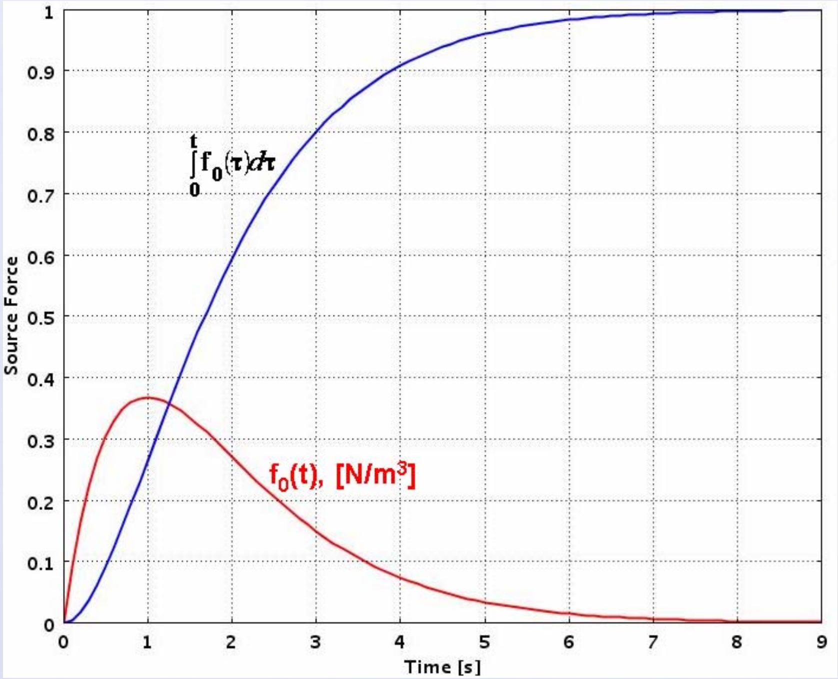
$$R(\tau) = \begin{cases} 0, \quad \tau < 1/\delta \\ \dfrac{\tau}{16\sqrt{6}}\left\{\begin{aligned} &6\mathrm{K}(k) - 18\Pi(8k^2,k) + (6-4\sqrt{3})\Pi\left[(20-12\sqrt{3})k^2,k\right] \\ &\qquad + (6+4\sqrt{3})\Pi\left[(20+12\sqrt{3})k^2,k\right] \end{aligned}\right\}, \quad 1/\delta < \tau < 1 \\ \dfrac{\tau/k}{16\sqrt{6}}\left\{\begin{aligned} &6\mathrm{K}\left(\frac{1}{k}\right) - 18\Pi\left(8,\frac{1}{k}\right) + (6-4\sqrt{3})\Pi\left[(20-12\sqrt{3}),\frac{1}{k}\right] \\ &\qquad + (6+4\sqrt{3})\Pi\left[(20+12\sqrt{3}),\frac{1}{k}\right] \end{aligned}\right\}, \quad 1 < \tau < \gamma \\ \text{Preceding} + \dfrac{\pi\tau}{24}(\tau^2-\gamma^2)^{-1/2}, \quad \tau > \gamma \end{cases}$$

Point Force Solution – 3D

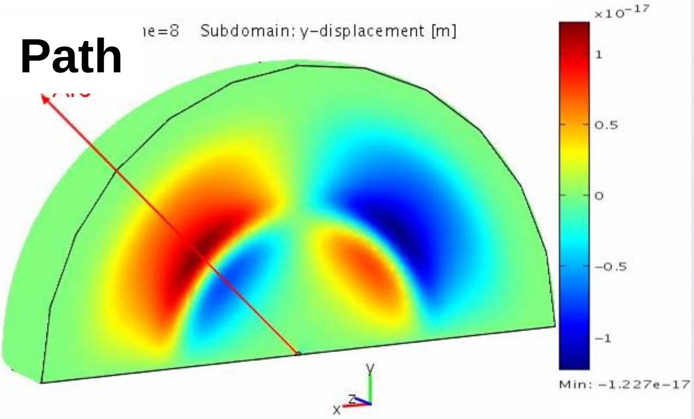


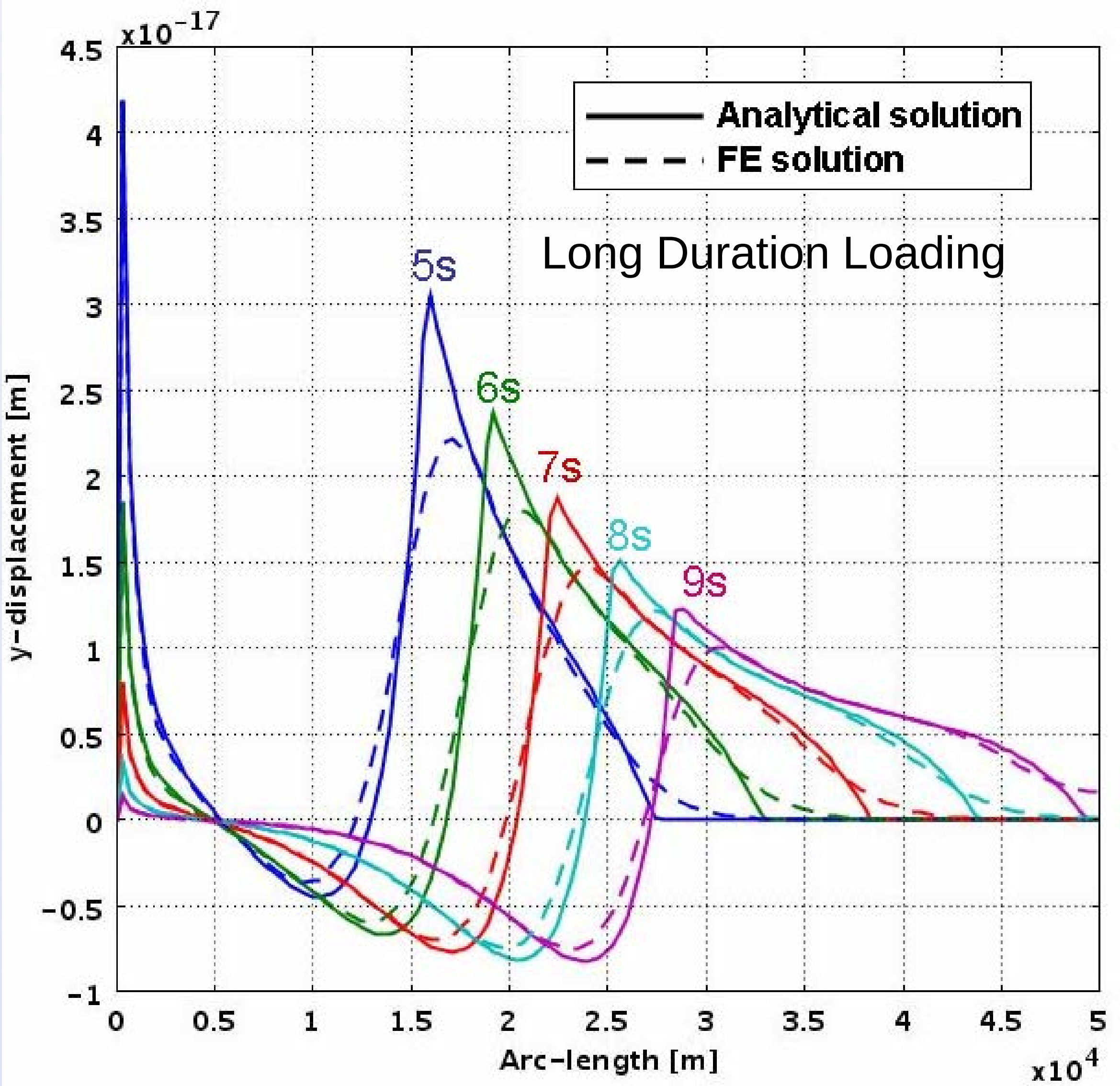
Quarter Symmetric Model
N_{DOF} = 574,000

Long Duration Loading



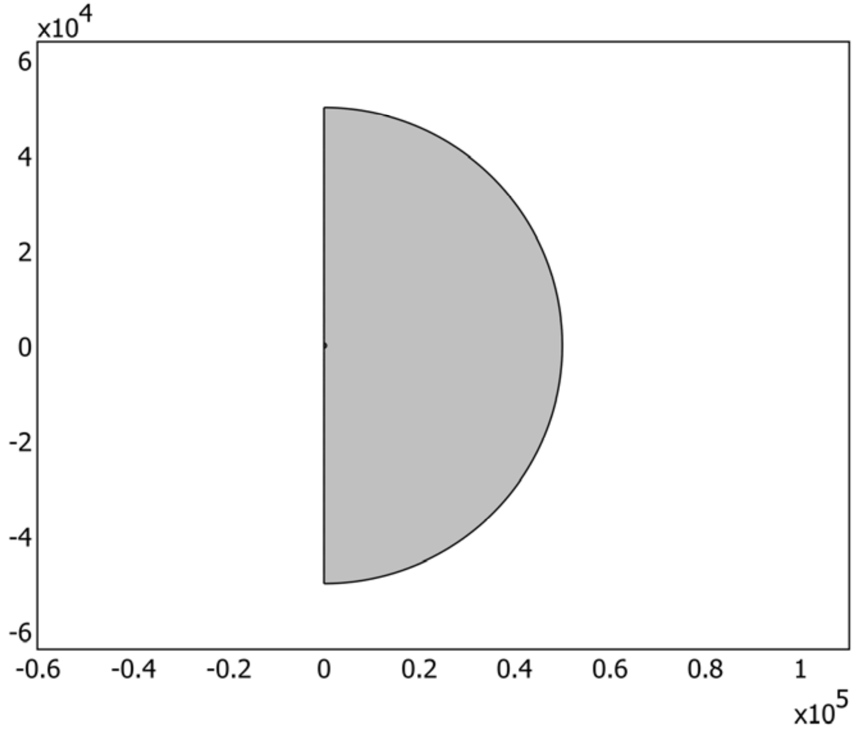
Comparison to Analytical Solution – 3D



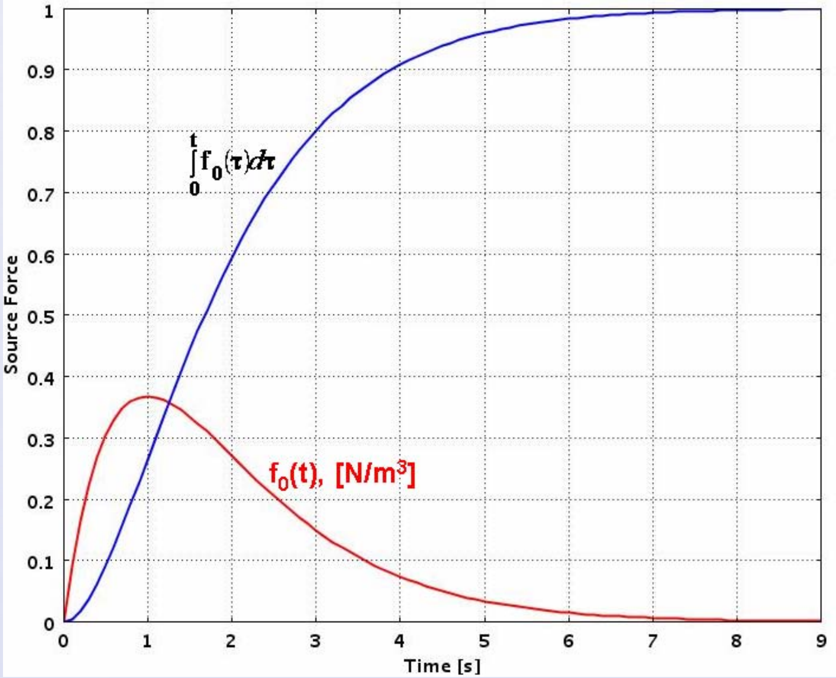


Point Force Solution – 2D

- Axisymmetric
- Plane Strain

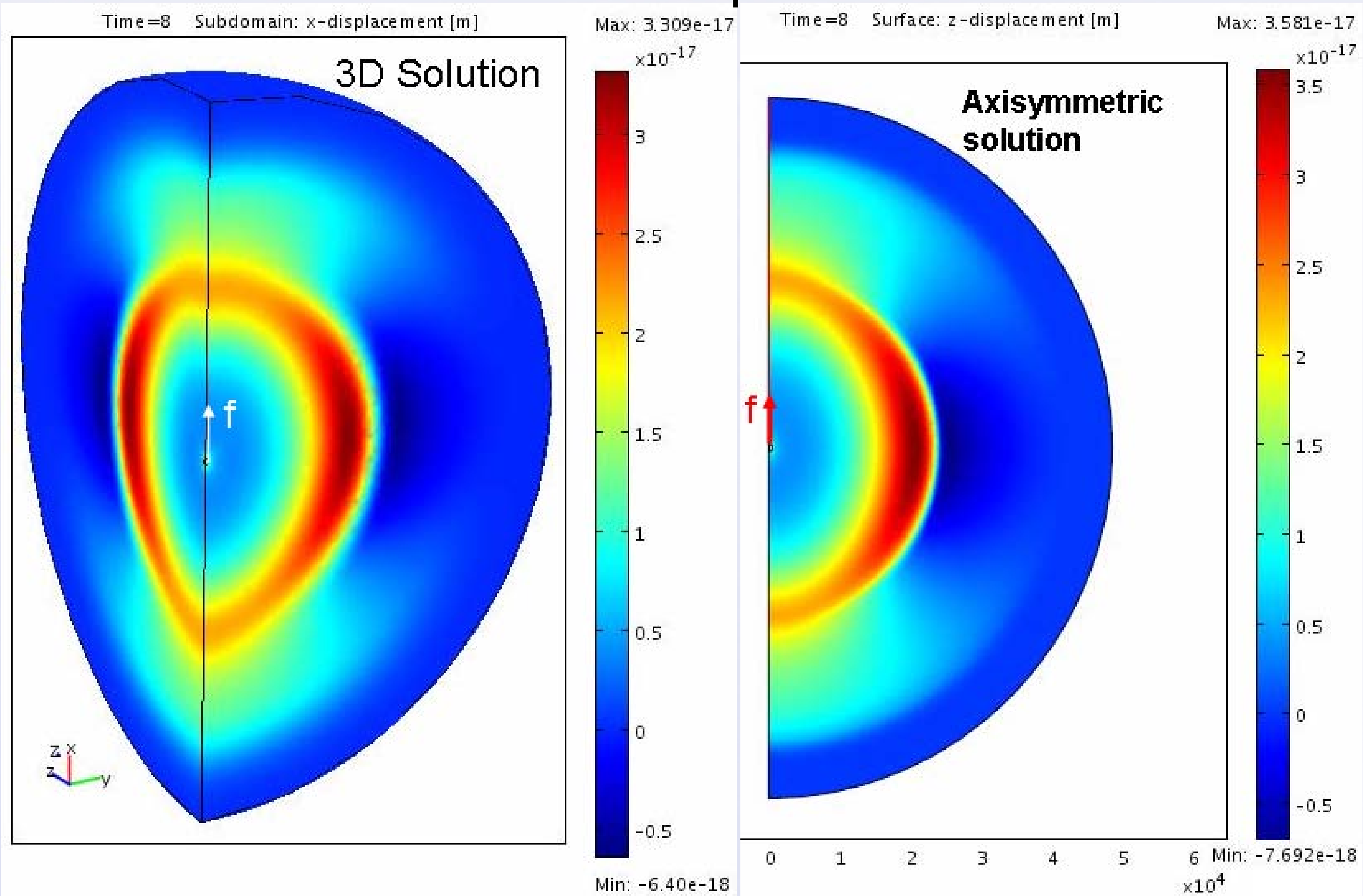


2D Model
N_{DOF} = 195,000

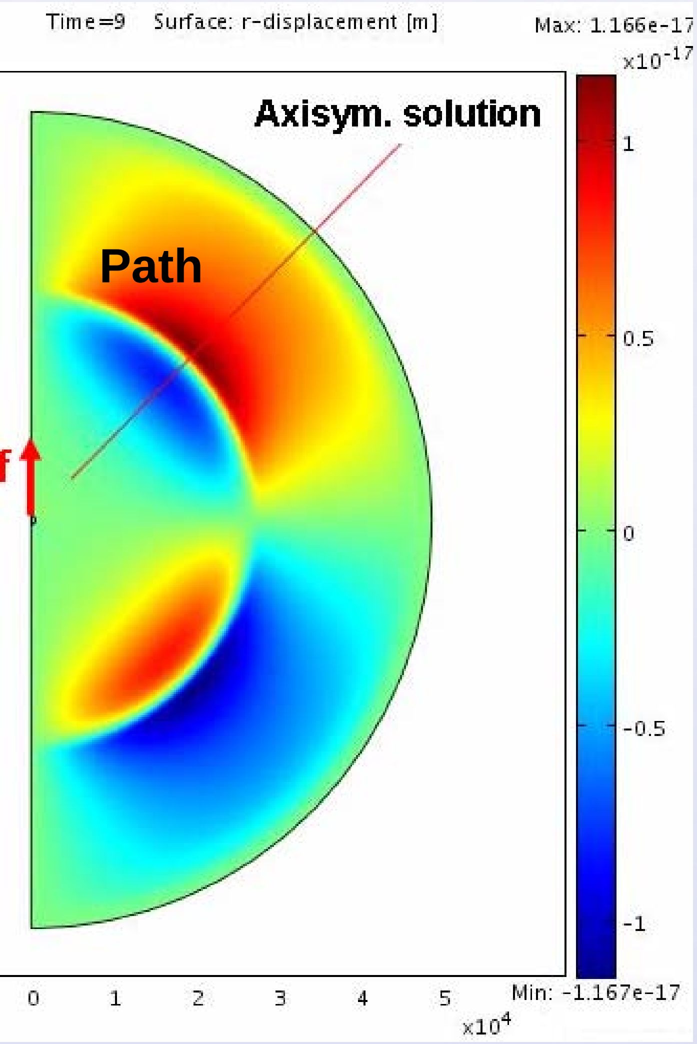


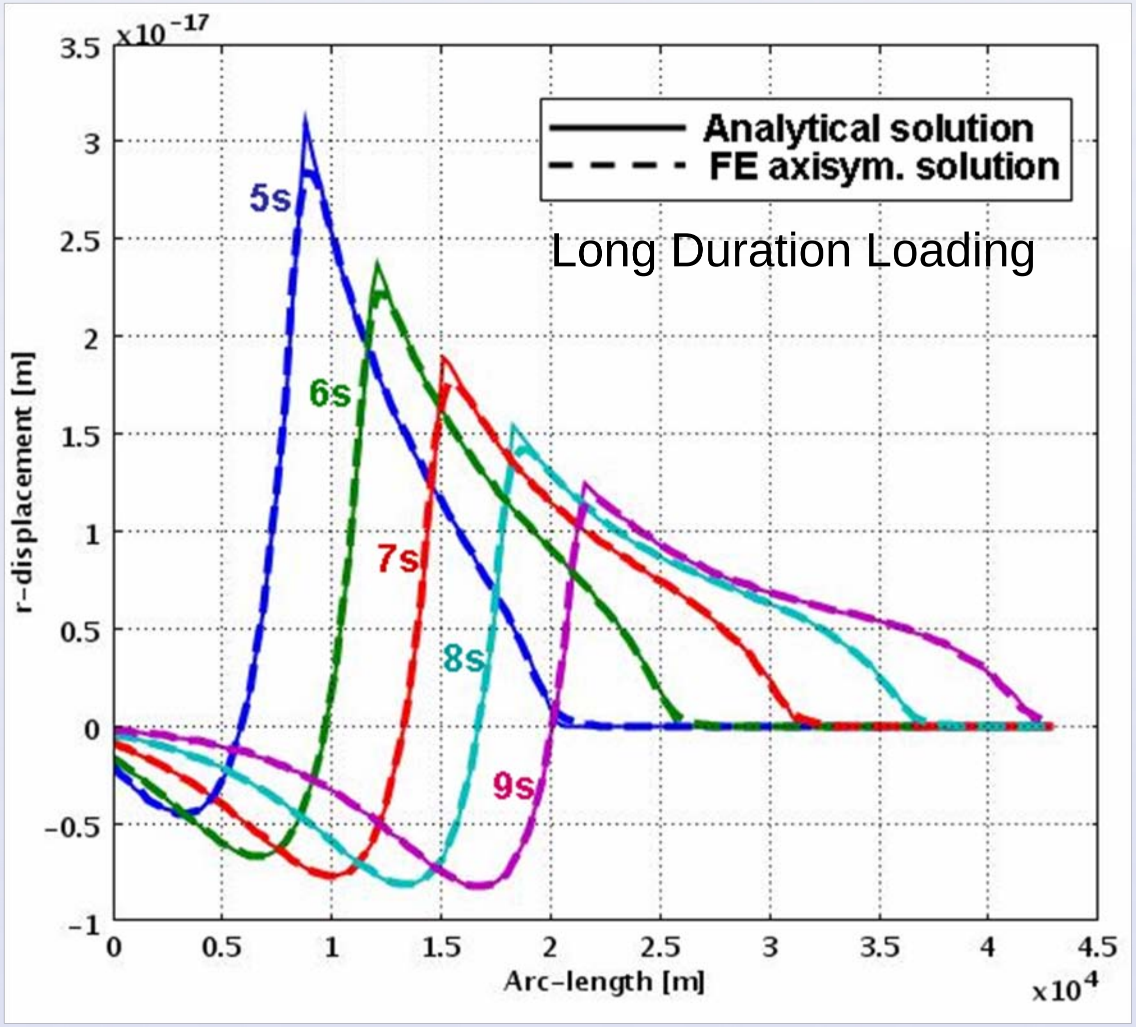
Comparison of 3D and Axisymmetric

Vertical Displacement

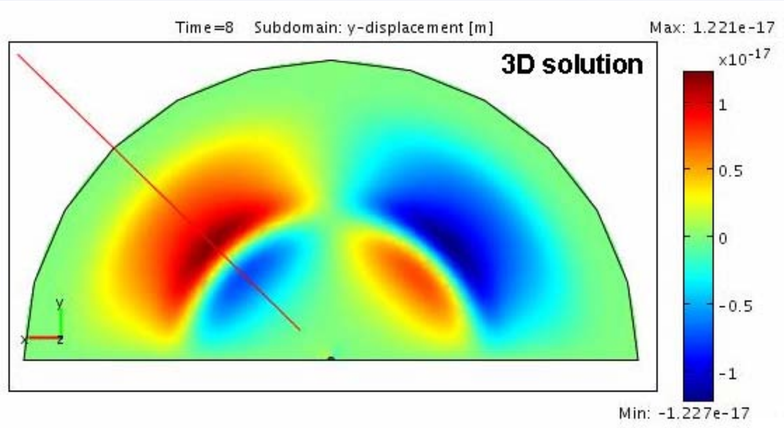


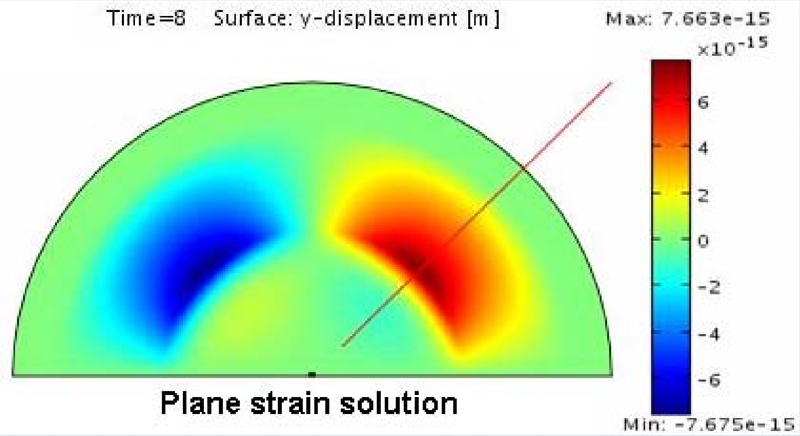
Comparison to Analytical Solution - Axisymmetric

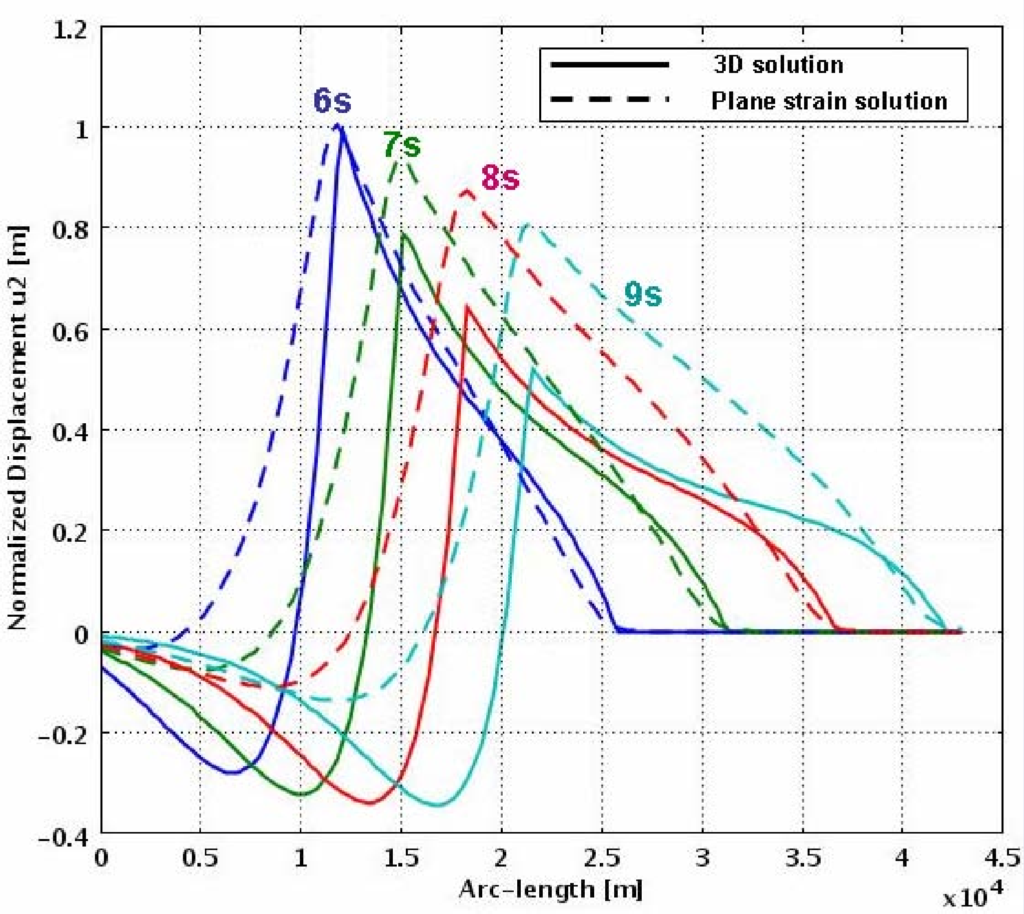




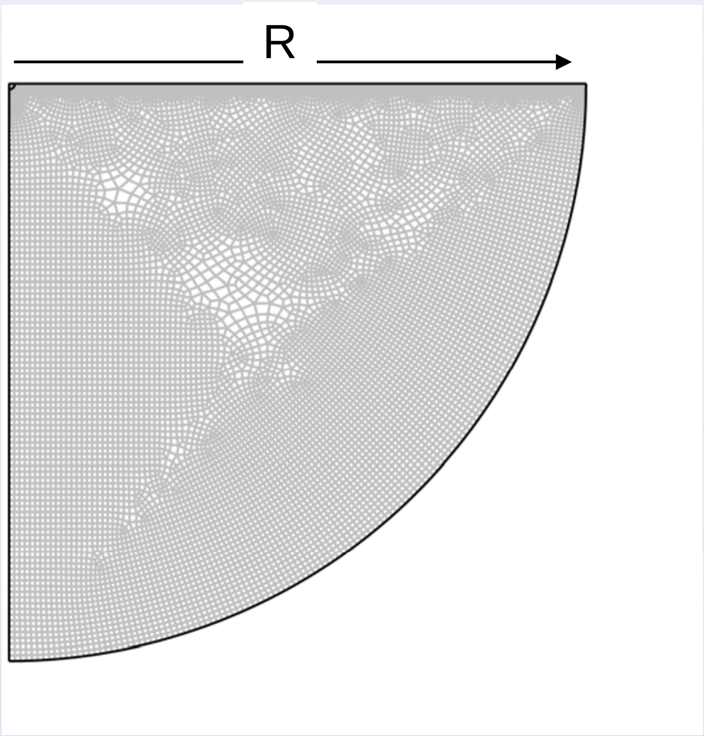
Comparison of 3D and Plane Strain





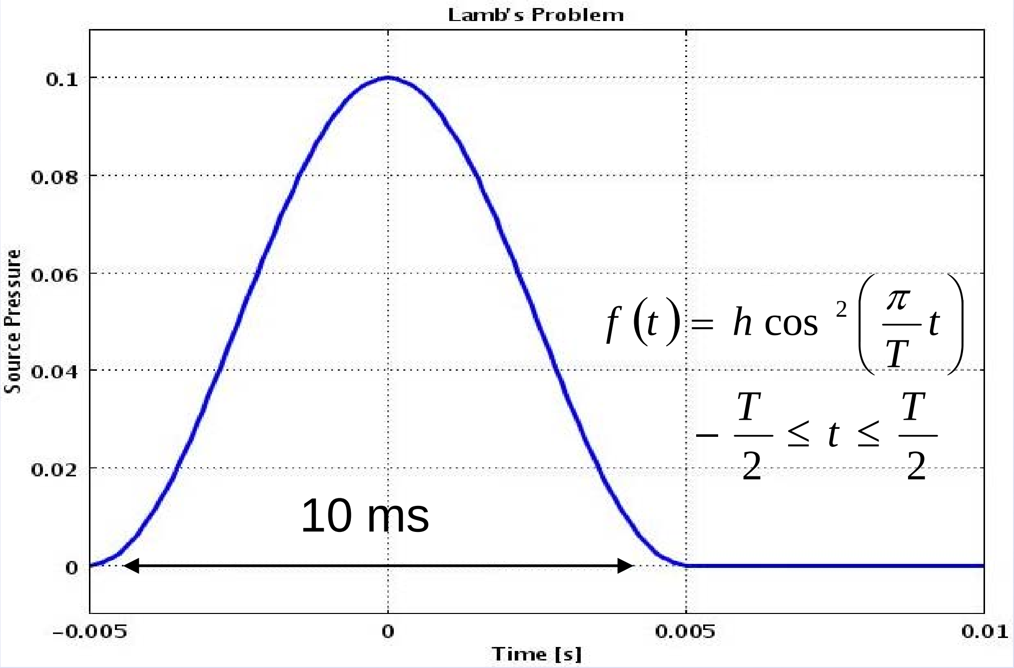


Surface Wave Problem

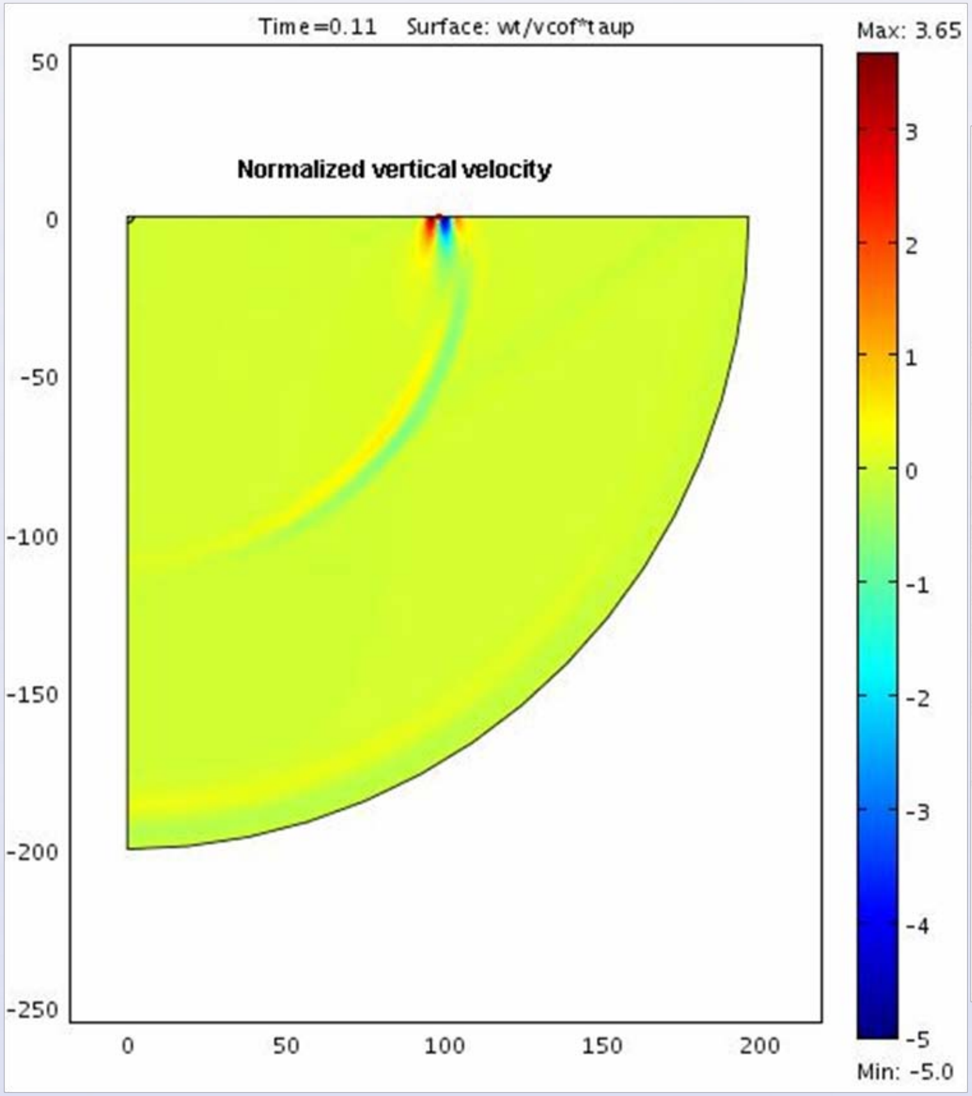


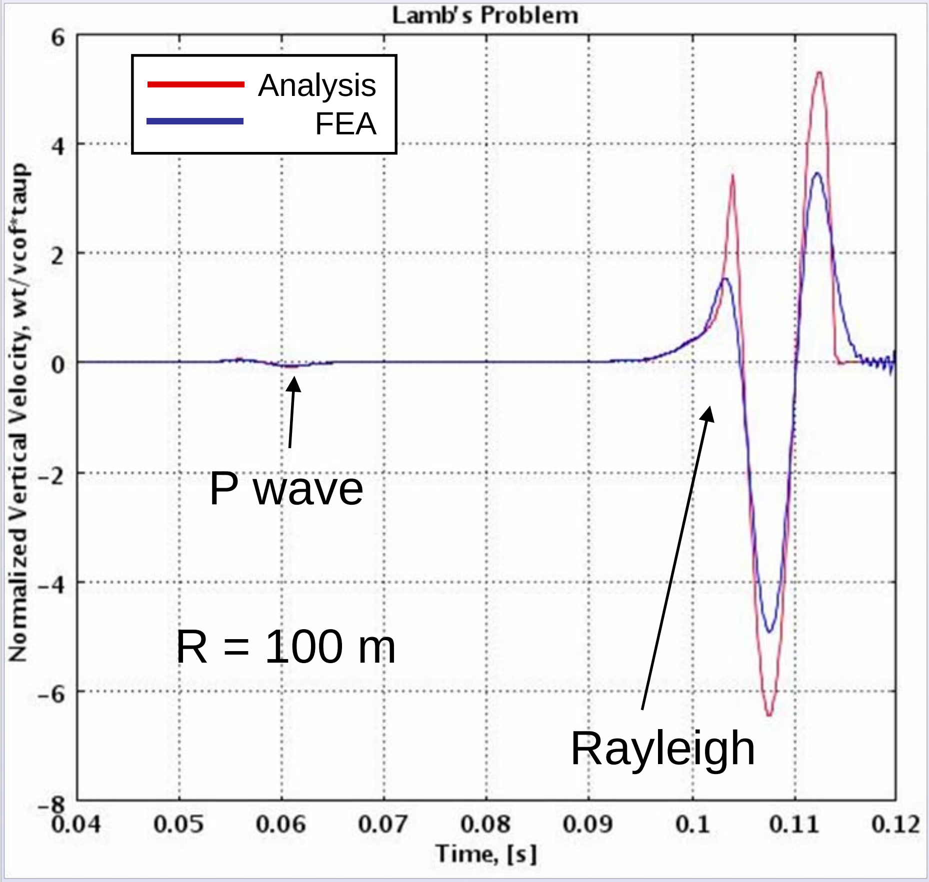
2D Model
R = 200 m
N_{DOF} = 23,000

Short Duration Loading

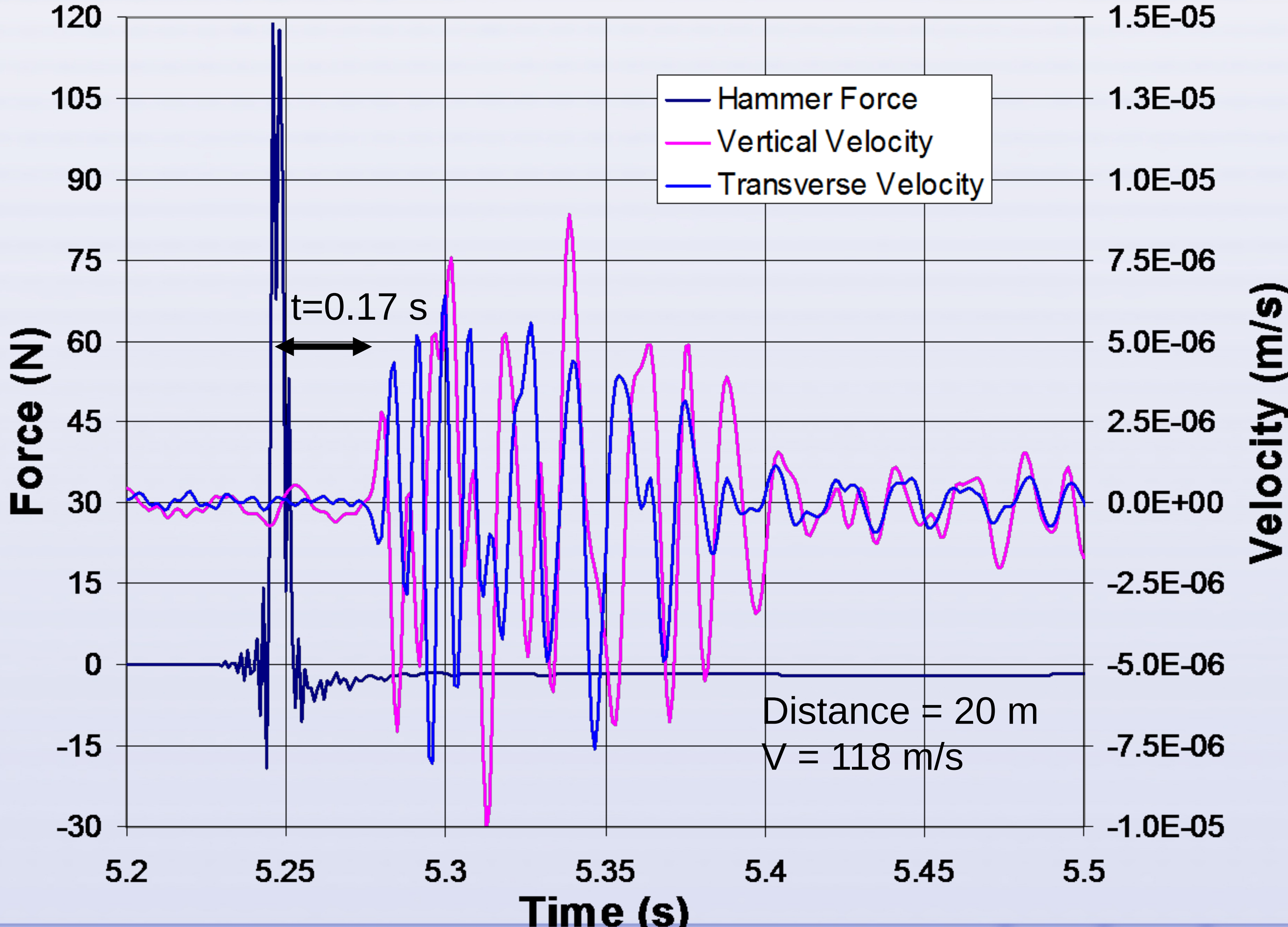


Surface Wave Problem – Vertical Velocity



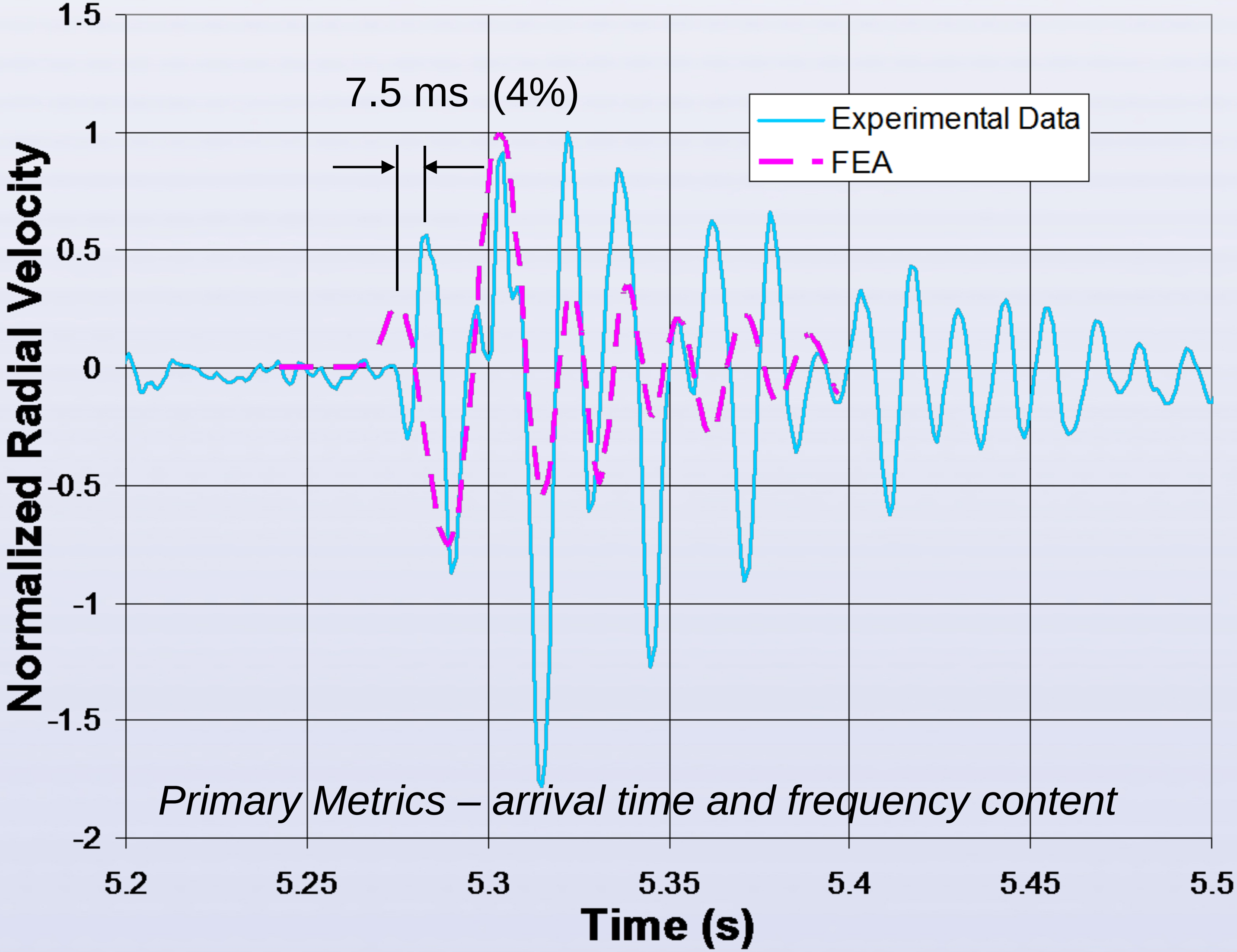


Hammer Data



120
105
90
75
60
45
30
15
0
-15
-30
1.5E-05
1.3E-05
1.0E-05
7.5E-06
5.0E-06
2.5E-06
0.0E+00
-2.5E-06
-5.0E-06
-7.5E-06
-1.0E-05
Force (N)
Velocity (m/s)
5.2
5.25
5.3
5.35
5.4
5.45
5.5
Time (s)
Hammer Force
Vertical Velocity
Transverse Velocity
t=0.17 s
Distance = 20 m
V = 118 m/s

Comparison w/ Exp Data



Summary

- Closed-form solution developed for
  - Elastic wave in infinite media
  - Elastic wave in semi-infinite media
- Computation models developed using Solid Mechanics Module
  - Three-dimensional
  - Two-dimensional
    - Axisymmetric
    - Plane Strain – not sufficiently accurate for point source
- Comparison with analytical solutions and experimental data
  - Agreement with arrival time, and frequency content

Conclusions

- COMSOL Multiphysics provides a sufficient level of accuracy for the problems of interest
- COMSOL Multiphysics provides a commercially available tool that can solve wave propagation problems on desktop computing resources